Analysis of hysteretic behaviour of a hillslope-storage kinematic wave model for subsurface flow

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Abstract

The objective of this work is to analyse the storage–flux hysteretic behaviour of a simplified model for subsurface flow processes. The subsurface flow dynamics is analysed by means of a model based on the kinematic wave assumptions and by using a width weighting/depth averaging scheme which allows to map the three-dimensional soil mantle into a one-dimensional profile. Continuity and a kinematic form of Darcy’s law lead to a hillslope-storage kinematic wave equation for subsurface flow, solvable with the method of characteristics. Adopting a second order polynomial function to describe the bedrock slope and an exponential function to describe the variation of the width of the hillslope with hillslope distance, we derive general solutions to the hillslope-storage kinematic wave equations, applicable to a wide range of hillslopes. These solutions provide a physical basis for deriving two geometric parameters \( a \) and \( w \) which define the hydrological similarity between hillslopes with respect to their characteristic response and hysteresis. The hysteresis \( \eta \), quantified by the area of the hysteretic dimensionless loop, has been therefore computed for a range of values of parameters \( a \) and \( w \).

Slopes exhibit generally clockwise hysteretic loop in the flux-storage plot, with higher groundwater mean volume for given discharge on rising limb than at same discharge on falling limb. It has been found that hysteresis increases with decreasing \( a \) and \( w \), i.e. with increasing convergence (for the shape) and concavity (for the profile), and vice versa. For relatively large values of \( a \) and \( w \) the hysteresis may take a complex pattern, with combination of clockwise to anticlockwise loop cycles. Application of the theory to three hillslopes in the Eastern Italian Alps provides an opportunity to examine how natural topographies are represented by the two hillslope hydrological similarity parameters.

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1. Introduction

Hysteresis is a common feature in several hydrological processes. It may occur in many input–output relationships between time-dependent quantities. In general, an input–output relationship is said to present hysteresis if the value of the output at a generic time does not only depend on the value of the input at the same instant, but also on the history of the input. As such, hysteresis identifies a class of strongly non linear processes, i.e. linearisation cannot encapsulate the observed phenomena [13,22].

In hydrology, the hysteretic behaviour of the relation between water retention and soil moisture tension has been studied for 70 years, dating back to the studies of Haynes [14] and Richards [24]. In applying the Darcy–Richards equation it was found that the soil moisture characteristic curves that link soil moisture content, soil water potential and hydraulic conductivity, were not single valued but were rather different for wetting and drying.

Hysteresis has also been found to emerge at various space and time scales and in many different hydrological processes. Observations of hysteresis in the storage-surface discharge relationships have been reported by a number of authors. For instance, Myrabø [19] reported distinct hysteretic effects in discharge-groundwater level relationships in a number of small catchments in Norway. This author showed that the behaviour of the discharge-groundwater relationship...
relationship is dependent on location, antecedent conditions and size of response area before each precipitation event. Hysteretic relationships between streamflow and groundwater level were identified also by Kendall et al. [18] for headwater catchments in Vermont (USA). Clockwise hysteresis (higher groundwater level for given discharge on rising limb than at same discharge on falling limb) was reported for riparian sites, suggesting that the riparian zone was the dominant source area during the rising limb of the melt hydrograph. Hysteresis was counter-clockwise at hillslope sites, suggesting that hillslope drainage controlled the snowmelt recession. Similar relationships were reported by Schumann and Herrmann [25] for two small mountainous basins in Germany.

Ewen and Birkinshaw [11] examined hysteretic loops in storage-discharge plots based on runoff data from the Slapton Wood catchment in UK. They identified a basic pattern of hysteretic loops, with higher groundwater mean volume for given discharge on rising limb than at same discharge on falling limb. Beven [4] reported hysteretic behaviour associated with individual storms in plots of relative storage vs relative runoff for a number of small catchments in UK.

A whole range of hydrological processes can either collectively or individually contribute to generate hysteresis at the catchment scale. For instance, Beven [4] mentioned “…the small scale matrix soil characteristics, the possibility of changing vertical and downslope connectivities of flow pathways as the soil dries and rewets, the development of the inter-unit patterns of antecedent wetness, …, the dynamics of contributing areas”.

Chirico [8] and Niedzialek and Ogden [21], in separate studies focused on the saturation excess runoff generation mechanism, examined the relationship between small catchment properties and the temporal growth and decay of saturated source areas. These authors used different physically based hydrologic models and were able to show that the temporal evolution of the extent of saturated source area versus catchment average soil water content during a number of wetting and drying cycles exhibits a wide variety of trajectories or hysteretic loops.

Chapman and Ong [6] reported an hysteresis loop between dimensionless flow and mean flow depth from shallow groundwater, for rising and falling conditions by applying a shallow groundwater equation for flow over a convexo-concave continuous curvilinear bed.

These findings support the view that hysteresis in the relation between average saturated soil water content and hillslope subsurface flow may contribute to hysteresis emerging at the hillslope and at the catchment scale and that hillslope geometry (together with other factors) may influence the structural characteristics of hysteresis.

The purpose of this paper is to characterise the hysteretic behaviour in the saturated storage–flux relationships at the hillslope scale. The subsurface flow dynamics is analysed by means of a simplified model based on the kinematic wave assumptions and using a method first proposed by Horton [17] and then further developed by Fan and Bras [12] to transfer the three-dimensional soil mantle into a one-dimensional profile. In particular, we explore the influence of topographic factors on the hysteretic behaviour.

The hillslope-storage subsurface flow equation considered in this paper takes into account the topographic controls exerted on the flow processes by the plan shape and profile curvature. Adopting a second order polynomial function to describe the bedrock slope and an exponential function to describe the variation of the width of the hillslope with hillslope distance, we derive general solutions to the hillslope-storage kinematic wave equations, applicable to a wide range of hillslopes. These analytical solutions afford a broad and clear view of the essential features of the problem and allows to derive two geometric parameters $\alpha$ and $\psi$ which define the hydrological similarity between hillslopes with respect to their characteristic response and hysteresis. Therefore, each hillslope may be represented in a $\alpha$–$\psi$ plot which allows deriving essential information on its response.

The paper is organised in the following way. Section 2 describes the hillslope-storage kinematic wave model used in the study and its analytical solutions. Section 3 reports results on the hysteretic behaviour for nine basic hillslope types; this section introduces the hillslope similarity parameters $\alpha$ and $\psi$, and describes their use to quantify the hysteretic effect. Section 4 describes the application of the hillslope similarity parameters to three hillslopes in the Eastern Italian Alps. Section 5 completes the paper with discussion and conclusions.

2. The hillslope-storage kinematic wave model

The saturated component of the hillslope response has traditionally been studied by means of the hydraulic groundwater theory. To overcome difficulties associated with three-dimensional models, a series of low-dimensional hillslope models have been recently developed [26,27]. These models are able to treat geometric complexity in a simple way by using a width weighting and depth averaging scheme which allows to map the three-dimensional soil mantle into a one-dimensional profile, resulting in a significant reduction in model complexity. This type of width weighting actually goes back to Horton [17] and since then was further developed by Beven [1] and by Fan and Bras [12]. One should note that these models are limited to the saturated component of the hillslope response and does not include the unsaturated component.

Fan and Bras [12] introduced the soil moisture storage function $S(x, t)$,

$$S(x, t) = w(x) h(x, t) \epsilon$$  \hspace{1cm} (1)

where $w(x)$ is the width of the hillslope at flow distance $x$, measured along the horizontal, from the divide (the so-called hillslope width function), $h(x)$ is the width-averaged water table height (measured along the vertical) at flow dis-
tance \(x, \varepsilon\) is drainable porosity and \(t\) is time (see also Fig. 1 for a graphical definition of the width and soil depth function). This allows reformulating the three-dimensional flow problem as a one-dimensional flow problem. The propagation of soil moisture storage in space and time, \(S(x, t)\) is constrained by the continuity equation and Darcy’s law. Along the hillslope the continuity equation reads

\[
\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = r(t)w(x)
\]

(2)

where \(r(t)\) is the recharge to the saturated layer and \(Q\) is the flow rate. Let us further assume that the flow rate \(Q\) is related to the storage \(S(x, t)\) through a kinematic form of Darcy’s equation

\[
Q = -k\frac{S}{\varepsilon} \frac{dz}{dx}
\]

(3)

where \(z\) is the elevation of the bedrock above a given datum.

Combining (3) and (1) and assuming no spatial variability in \(k\) and \(\varepsilon\) provides a quasi-linear wave equation in terms of soil moisture storage

\[
b(x) \frac{\partial S}{\partial x} + \frac{\partial S}{\partial t} = c(x, S)
\]

(4)

where

\[
b(x) = -\frac{kz'(x)}{\varepsilon}
\]

(5)

and

\[c(x, S) = rw(x) + \frac{kz''(x)}{\varepsilon} S
\]

(6)

and \(z'\) and \(z''\) are first and second derivatives of the bedrock profile curvature function \(z(x)\) with respect to \(x\).

In the following we will use the profile curvature function given by Fan and Bras [12]

\[z(x) = H + \beta x + \gamma x^2\]

(7)

In Eq. (7), values of \(\beta\) are always negative, values of \(\gamma > 0\) define concave profiles, \(\gamma < 0\) define convex profiles, and for \(\gamma = 0\) the profile is linear. A potential limitation of Eq. (7) is that it cannot describe a convexo-concave hillslope profile. Research is on going toward identifying a proper S-shaped function that can be integrated out of (5) and (6). Furthermore, we will use an exponential formulation for the width function, defined as [27]

\[w(x) = C \exp(ax)
\]

(8)

In Eq. (8), \(C\) defines the width of the hillslope at the divide, and \(a\) represents the degree of convergence. Values of \(a > 0\) define divergent shapes, \(a < 0\) define convergent shapes, and for \(a = 0\) the shape is rectangular.

Because we assume that the kinematic wave approximation holds, the analysis presented here is limited to moderate to steep slopes. The lower limit of relevant slopes is defined by the ratio of gravity drainage versus diffusion drainage [2], as well as by the ratio of water table gradients and bedrock gradients [16].

Since we have formulated the kinematic form of Darcy’s law under the assumption of horizontal flow lines, there exists also an upper limit to the relevant bedrock slopes [7]. However, Hilberts et al. [16] have reported good accuracy for the kinematic wave model when applied to steep hillslopes with gradients of 30°.

2.1. Analytical solution using the method of characteristics

With the method of characteristics, Eq. (4) can be written as two ordinary differential equations:

\[
\frac{dx}{dt} = b(x) = -\frac{kz'}{\varepsilon} - \frac{k(\beta + 2\gamma x)}{\varepsilon}
\]

(9)

\[
\frac{dS}{dx} = \frac{c(x, S)}{b(x)} = -\frac{\varepsilon}{kz^2} rw - \frac{z''}{z'} S
\]

(10)

Eq. (9) describes a family of curves in the \((x, t)\) plane, called the characteristic curves, and (10) describes how the soil moisture wave, \(S\), propagates along each curve. If the recharge rate \(r(t)\) is assumed constant during the recharge (whose duration is indicated by \(T_c\), we can obtain an analytical solution for the storage \(S\). We assume the following initial and boundary Dirichlet conditions:

\[S(x, 0) = g(x) \quad 0 \leq x \leq L; \quad \text{with } g(0) = 0\]

\[S(0, t) = g(0) = 0, \quad \forall t\]

(11)

(12)

where \(L\) defines the length of the hillslope in the horizontal direction. Since the governing Eq. (4) is of first order only, the kinematic waves possess only one system of characteristics. This would also imply that kinematic waves travel in the downstream direction only, and that the solution of the equation does not require a downstream boundary condition. As a result, when the porous medium is cut at \(x = L\) by the bed of a stream or lake, it will overpredict the length of the seepage face at the downstream boundary, since it takes no account of downstream effects in the vicinity of the boundary [2].

Fig. 1. Three-dimensional view of a convergent hillslope.
The response time of the hillslope is given by

\[
T_e = -\frac{\varepsilon}{2\gamma k} \ln \left( 1 + \frac{2\gamma L}{\beta} \right) \text{ for } \gamma \neq 0 \\
T_e = -\frac{L \varepsilon}{\beta k} \text{ for } \gamma = 0
\]

For \( \gamma \neq 0 \), the structure of this expression leads to the following geometric constraint

\[
1 + \frac{2\gamma L}{\beta} > 0
\]

Since \( \beta \) is always negative, for convex hillslopes (\( \gamma < 0 \)) this inequality is always true, so there is no further constraint on \( \beta, \gamma \) or \( L \). On the contrary, for concave hillslopes (\( \gamma > 0 \)), the constraint on the curvature is

\[
\gamma < -\frac{\beta}{2L}.
\]

The condition \( \gamma = -\frac{\beta}{2L} \) represents a situation with horizontal bed for \( x = L \). Note that the solutions investigated here are for absence of surface saturation. This means that the soil moisture capacity always exceeds the computed soil moisture. Analysis of hysteretic behaviour of the relationship between the saturated area and average hillslope soil moisture [8,21] is beyond the scope of this paper and will be presented in future papers.

A condition of partial equilibrium is investigated and three domains are distinguished for soil moisture storage at \( x = L \), as follows:

for \( 0 \leq t \leq T_r \):

\[
S(L, t) = -\frac{\varepsilon r}{k} \left[ e^L - A(\xi) \right] + g(\xi) \exp \left( \frac{2\gamma k}{\varepsilon} t \right) \text{ for } \gamma \neq 0 \\
S(L, t) = -\frac{\varepsilon r}{k} \left[ e^L - A(\xi) \right] + g(\xi) \text{ for } \gamma = 0
\]

where

\[
\xi = \left( L + \frac{\beta}{2\gamma} \right) \exp \left( \frac{2\gamma k}{\varepsilon} t \right) - \frac{\beta}{2\gamma} \text{ for } \gamma \neq 0 \\
\xi = L + \frac{k \beta}{\varepsilon} (t - T_r) \text{ for } \gamma = 0
\]

and \( A(x) \) is the uphill drainage area at location \( x \).

for \( T_r \leq t \leq T_c \):

\[
S(L, t) = S(x_0) \frac{\beta + 2\gamma x_0}{\beta + 2\gamma L} \text{ for } \gamma \neq 0 \\
S(L, t) = S(x_0) \text{ for } \gamma = 0
\]

where

\[
x_0 = \left( L + \frac{\beta}{2\gamma} \right) \exp \left( \frac{2\gamma k}{\varepsilon} (t - T_r) \right) - \frac{\beta}{2\gamma} \text{ for } \gamma \neq 0 \\
x_0 = L + \frac{k \beta}{\varepsilon} (t - T_r) \text{ for } \gamma = 0
\]

and

\[
S(x_0, t) = -\frac{\varepsilon r}{k} \left[ A(x_0) - A(\xi) \right] + g(\xi) \exp \left( \frac{2\gamma k}{\varepsilon} T_r \right) \text{ for } \gamma \neq 0 \\
S(x_0, t) = -\frac{\varepsilon r}{k} \left[ A(x_0) - A(\xi) \right] + g(\xi) \text{ for } \gamma = 0
\]

for \( T_e \leq t \leq T_e + T_r \):

\[
S(L, t) = S(x_0) \frac{\beta + 2\gamma x_0}{\beta + 2\gamma L} \text{ for } \gamma \neq 0 \\
S(L, t) = S(x_0) \text{ for } \gamma = 0
\]

where

\[
x_0 = \left( L + \frac{\beta}{2\gamma} \right) \exp \left( \frac{2\gamma k}{\varepsilon} (t - T_r) \right) - \frac{\beta}{2\gamma} \text{ for } \gamma \neq 0 \\
x_0 = L + \frac{k \beta}{\varepsilon} (t - T_r) \text{ for } \gamma = 0
\]

2.2. Determination of subsurface discharge at the hillslope outlet

The subsurface flow rate \( Q(L, t) \) at the hillslope outlet may be obtained by combining Eq. (3) and Eqs. (13), (15) and (18). When the exponential formulation of Eq. (8) is used to describe the width function, the following solutions are obtained:

for \( 0 \leq t \leq T_r \):

\[
Q(L, t) = -\frac{k}{\varepsilon} \left( \beta + 2\gamma L \right) \left[ g(\xi) \exp \left( \frac{2\gamma k}{\varepsilon} t \right) \right. \\
\left. - \frac{\varepsilon r c}{k a} \exp \left( a L - \exp \left( a \left( L + \frac{k \beta}{\varepsilon} \right) \right) \right) \right] \text{ for } \gamma \neq 0 \\
Q(L, t) = r \left[ \frac{c}{a} \left( \exp(aL) - \exp \left( a \left( L + \frac{k \beta}{\varepsilon} \right) \right) \right) \right] + g(\xi) \text{ for } \gamma = 0
\]

for \( g(\xi) = 0 \) one obtains

\[
Q(L, t) = -\frac{c}{a} \left( \exp(aL) - \exp \left( a \left( L + \frac{k \beta}{\varepsilon} \right) \right) \right) \text{ for } \gamma \neq 0 \\
Q(L, t) = r \left[ \frac{c}{a} \left( \exp(aL) - \exp \left( a \left( L + \frac{k \beta}{\varepsilon} \right) \right) \right) \right] \text{ for } \gamma = 0
\]
for \( g(\xi) = 0 \) one obtains

\[
Q(L, t) = \frac{rc}{a} \left( \exp \left( a \left( \left( \frac{\beta}{2\gamma} + L \right) \exp \left( \frac{2\beta k}{\gamma} (t - T_r) \right) \right) - \frac{\beta}{2\gamma} \right) - 1 \right) \quad \gamma \neq 0
\]

\[
Q(L, t) = r \left[ \frac{c}{a} \left( \exp \left( a \left( L + \frac{k\beta}{\gamma} (t - T_r) \right) \right) \right) \right] \quad \gamma = 0
\]  

(24)

for \( T_r \leq t \leq T_c + T_r \)

\[
Q(L, t) = \frac{rc}{a} \left( \exp \left( a \left( \left( \frac{\beta}{2\gamma} + L \right) \exp \left( \frac{2\beta k}{\gamma} (t - T_r) \right) \right) - \frac{\beta}{2\gamma} \right) - 1 \right) \quad \gamma \neq 0
\]

\[
Q(L, t) = r \left[ \frac{c}{a} \left( \exp \left( a \left( L + \frac{k\beta}{\gamma} (t - T_r) \right) \right) \right) \right] \quad \gamma = 0
\]  

(25)

which is obviously independent from initial conditions.

### 2.3. Determination of the hillslope saturated moisture content

The hillslope saturated moisture content at time \( t, V(t) \), may be obtained by applying the mass conservation equation over the hillslope. However, the general solution for the hillslope saturated moisture content involves non-elementary integrals (i.e., there is no explicit formula in terms of elementary functions for the antiderivative). Owing to this reason, analytical solutions are provided here for two elementary cases: (i) \( a = 0 \), which represents a rectangular hillslope with any profile that conforms to Eq. (7), and (ii) \( \gamma = 0 \), which represents a planar hillslope with any planform that conforms to Eq. (8). These solutions are as follows:

for \( 0 \leq t \leq T_r \)

\[
a = 0 : \quad V(Q(L, t)) = V(0) - \frac{e}{2\gamma k} Q(L, t) - \frac{rc}{2\gamma} t
\]

\[
\gamma = 0 : \quad V(Q(L, t)) = V(0) - \frac{e}{a\beta k} Q(L, t) - \frac{rc}{a} t
\]  

(26)

for \( T_r \leq t \leq T_c \)

\[
a = 0 : \quad V(Q(L, t)) = V(T_c) - \frac{e}{2\gamma k} [Q(L, t) - Q(L, T_r)]
\]

\[
\gamma = 0 : \quad V(Q(L, t)) = V(T_c) - \frac{e}{a\beta k} [Q(L, t) - Q(L, T_c)] + \frac{rc}{a} (t - T_c)
\]  

(27)

for \( T_c \leq t \leq T_c + T_r \)

\[
a = 0 : \quad V(Q(L, t)) = V(T_c) - \frac{e}{2\gamma k} [Q(L, t) - Q(L, T_c)] + \frac{rc}{a} (t - T_c)
\]

\[
\gamma = 0 : \quad V(Q(L, t)) = V(T_c) - \frac{e}{a\beta k} [Q(L, t) - Q(L, T_c)]
\]  

(28)

For the other cases, involving values of both \( a \) and \( \gamma \) different from zero, solutions will be provided in the following by means of numerical integration.

Inspection of the structure of Eqs. (26)–(28) shows that the maximum value for the hillslope saturated moisture content (\( V_{\text{max}} \)) corresponds to a recharge duration at least equal to the \( T_c \). It is interesting to note that the dimensionless hillslope saturated moisture content, obtained as \( \sigma(t) = V(t)/V_{\text{max}} \), is function of the geometric parameters and of the dimensionless time \( \tau = t/T_c \), and outflow \( \phi(t) = Q(L, t)/Q_{\text{max}} \), as follows (for the sake of brevity, we reported only the relationships obtained by assuming \( T_r = T_c \)):

for \( 0 \leq \tau \leq 1 \)

\[
a = 0 : \quad \sigma(\tau) = \phi(\tau) \left( \frac{2\frac{\tau}{\beta}}{\frac{\tau}{\beta} - \ln \left( \frac{2\frac{\tau}{\beta}}{\beta} + 1 \right)} \right)
\]

\[
- \tau \left( \frac{\ln \left( \frac{2\frac{\tau}{\beta} + 1}{\frac{\tau}{\beta}} \right)}{\frac{\tau}{\beta} - \ln \left( \frac{2\frac{\tau}{\beta} + 1}{\frac{\tau}{\beta}} \right)} \right)
\]  

(29)

\[
\gamma = 0 : \quad \sigma(\tau) = \phi(\tau) \left( \frac{\exp(\alpha L) - 1}{\exp(\alpha L) - 1 - \alpha L} \right)
\]

\[
- \tau \left( \frac{\alpha L}{\exp(\alpha L) - 1 - \alpha L} \right)
\]
Table 1
Geometrical parameters for the nine slopes

<table>
<thead>
<tr>
<th>No.</th>
<th>Profile</th>
<th>Plan</th>
<th>$\beta$ (\textdegree)</th>
<th>$\gamma$ (m$^{-1}$)</th>
<th>$C$ (m)</th>
<th>$a$ (m$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Conc down</td>
<td>Divergent</td>
<td>-0.3</td>
<td>0.001</td>
<td>3</td>
<td>0.036</td>
</tr>
<tr>
<td>2</td>
<td>Straight</td>
<td>Divergent</td>
<td>-0.182</td>
<td>0</td>
<td>3</td>
<td>0.036</td>
</tr>
<tr>
<td>3</td>
<td>Conc up</td>
<td>Divergent</td>
<td>-0.3</td>
<td>-0.001</td>
<td>3</td>
<td>0.036</td>
</tr>
<tr>
<td>4</td>
<td>Conc down</td>
<td>Straight</td>
<td>-0.03</td>
<td>0.001</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>Straight</td>
<td>Straight</td>
<td>0.182</td>
<td>0</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>Conc up</td>
<td>Straight</td>
<td>-0.1</td>
<td>0</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>Conc down</td>
<td>Convergent</td>
<td>-0.03</td>
<td>0.001</td>
<td>120</td>
<td>-0.038</td>
</tr>
<tr>
<td>8</td>
<td>Straight</td>
<td>Convergent</td>
<td>-0.182</td>
<td>0</td>
<td>120</td>
<td>-0.038</td>
</tr>
<tr>
<td>9</td>
<td>Conc up</td>
<td>Convergent</td>
<td>-0.1</td>
<td>-0.001</td>
<td>120</td>
<td>-0.038</td>
</tr>
</tbody>
</table>

For $1 \leq \tau \leq 2$

$$a = 0: \quad \alpha(\tau) = 1 + (\phi(\tau) - 1) \left( \frac{2 \frac{aL}{\tau} - \ln \left( 2 \frac{aL}{\tau} + 1 \right)}{2 \frac{aL}{\tau} - \ln \left( 2 \frac{aL}{\tau} + 1 \right)} \right)$$

$$+ (\tau - 1) \left( \frac{\ln \left( 2 \frac{aL}{\tau} + 1 \right)}{2 \frac{aL}{\tau} - \ln \left( 2 \frac{aL}{\tau} + 1 \right)} \right)$$

$$\gamma = 0: \quad \alpha(\tau) = 1 + (\phi(\tau) - 1) \left( \frac{\text{exp}(aL) - 1}{\text{exp}(aL) - 1 - aL} \right)$$

$$+ (\tau - 1) \left( \frac{aL}{\text{exp}(aL) - 1 - aL} \right)$$

(30)

3. Analysis of the hysteretic behaviour for basic hillslope types

This section is devoted to the analysis of the relationship between the hillslope water content and the subsurface discharge at the base of the hillslope for various hillslope shapes. For this purpose, we analyse nine hillslopes defined by the parameters reported in Table 1. As suggested by Dikau [9] and following Troch et al. [26], these hillslopes are obtained by combining three plan and three profile curvatures, thus defining nine basic geometric relief forms (Fig. 2). We have assumed $L = 100$ m, $k = 1$ m h$^{-1}$, $\varepsilon = 0.3$, $r = 1$ mm h$^{-1}$ and dry initial conditions. With this choice of hydraulic and geometrical parameters, the nine hillslopes have the same response time $T_c$ (equal to 164.8 h) and same area (equal to 3000 m$^2$). The drop in elevation between divide and outlet is 20 m for cases 1, 3, 4, 6, 7 and 9; it is 18.2 m for cases 2, 5 and 8. For each hillslope, we examined response to 4 different recharge events, characterised by the same intensity and by duration equal to 0.25$T_c$, 0.50$T_c$, 0.75$T_c$, and 1.00$T_c$, respectively.

By using the analytical relationships presented in Section 2.3, and numerical integration for the cases with both $a$ and $\gamma$ different from zero, we derived the relationships between discharge and volume for the nine hillslopes. Figs. 3 and 4 summarise the main results from this experiment reporting the time series of outflow and soil moisture volume (Fig. 3) and the looped response trajectories in the storage–flux ($V$–$Q$) plot (Fig. 4) for the nine hillslopes. Fig. 5 describes the trajectories of the dimensionless variables $\phi(t)$ and $\alpha(t)$, for cases 1, 3, 7 and 9.

As expected, slope 5 shows the typical ramp response function for outflow. The diverging slopes (cases 1, 2 and 3) and, though to a lesser extent, even the straight and convex slope (case 6) have approximately a first-order step response function for subsurface drainage. This response is characterised by a quick rise at the beginning of the recharge event and a slow increase towards the end. The other hillslopes show exponential growth for outflow, with a slow increase at the beginning of the recharge event and a faster rise towards the end.

The hillslope saturated moisture content time series always display a pattern which approximates a first-order step response. This is very marked for diverging slopes and much less for converging slopes, with straight slopes exhibiting an intermediate behaviour. As a result, diverging slopes show an almost parallel pattern for outflow and saturated moisture content time series, both in the wetting and in the drying phase. This has a marked effect on both the maximum saturated moisture content and on the hysteretic behaviour, which decrease with increasing the degree of divergence. On the contrary, outflow and saturated moisture content exhibit a contrasting time pattern for converg-
Fig. 3. Time series of subsurface flow rates and hillslope saturated moisture content for the nine hillslopes.

Fig. 4. Storage-flux relationships for the nine hillslopes.
ing slopes. Outflow increases very slowly (quickly) at the onset (end) of the recharge event, whereas moisture tends to follow a more steady increase throughout the recharge. The different patterns in volume and discharge time series translate into very distinct response trajectories in the storage–flux \((V-Q)\) plot, with a marked hysteretic effect. This effect, as well as the maximum saturated moisture content, increases with increasing the degree of convergence.

In Fig. 4 all slopes but case 3 exhibit clockwise hysteretic loop in the \(V-Q\) plot. It is interesting to note that the clockwise loop patterns are consistent with those reported by Ewen and Birkinshaw [11] for storage-discharge plots based on runoff data from the Slapton Wood catchment in UK. However, it should be noted that catchment-scale hysteresis is controlled by the combination of hysteretic effects of hillslopes with different geometric configurations. Therefore, it is not easy to compare hysteretic effects observed at the catchment scale with those emerging at the hillslope scale. Only slope 3, characterised by a combination of gradual divergence and convex profile curvature, shows a partly anticlockwise hysteretic loop.

The hysteretic effect is controlled essentially by the different patterns of saturated soil moisture \(S(x,t)\) along the slope during the rising and falling limb. Furthermore, for a given value of \(S(L)\), i.e., for a given value of discharge at the outlet, the mean value of the corresponding saturated soil moisture pattern will be generally less during the falling limb than during the rising limb, thus giving rise to the clockwise hysteretic effect reported in Fig. 4.

Different trajectories are also associated to the response to recharge events with duration less than response time. As implied by Eq. (27), the relationship between discharge and volume is linear for cases 2, 4, 6 and 8, with a positive slope for cases 2 and 6, and a negative slope for cases 4 and 8. More generally, for hillslopes 1, 2, 3 and 6, the trajectory for the period comprised between the end of the recharge and the response time is associated to a decrease of both discharge and volume. For hillslope 5 the outflow stays constant during this period, as expected, while storage decreases. For remaining hillslopes 4, 7, 8 and 9, this trajectory shows a decrease of storage and an increase of discharge.

3.1. Dimensional inspection

This section aims to analyse the general character of topographic control on hysteretic behaviour. While in the previous section we analysed the hysteretic storage–flux relationships for specific hillslopes, our objective here is to derive geometrical similarity parameters which can be used to understand landscape topographic control on hysteresis. Two ubiquitous parameters found in the previous sections are

Fig. 5. Normalised storage \((\sigma(t) = V(t)/V_{\text{max}})\) versus flux \((\phi(t) = Q(L,t)/Q_{\text{max}})\) relationships for the hillslopes 1, 3, 7 and 9.
\[ \alpha = aL \]
\[ \psi = \ln \left( \frac{2\gamma L}{\beta} + 1 \right) \]

The divergency parameter \( \alpha \) is related to the plan shape of the hillslope: values of \( \alpha > 0 \) define divergent shapes, \( \alpha < 0 \) define convergent shapes, and for \( \alpha = 0 \) the shape is rectangular. The convexity parameter \( \psi \) is related to the curvature profile of the hillslope: values of \( \psi < 0 \) define concave profiles, \( \psi > 0 \) define convex profiles, and for \( \gamma = 0 \) the profile is linear.

Using these parameters and the dimensionless variables already introduced in Section 2.3, it is possible to write Eqs. (21)-(25) in a compact manner as follows:

for \( 0 \leq \tau \leq 1 \)
\[ \varphi(\tau) = \frac{1 - \exp \left( -\alpha \frac{1 - \exp(-\psi \tau)}{\exp(-\psi)} \right)}{1 - \exp(-\alpha)} \]  
for \( 1 \leq \tau \leq 2 \)
\[ \varphi(\tau) = \frac{\exp \left( -\alpha \frac{1 - \exp(-\psi(\tau - 1))}{\exp(-\psi)} \right) - \exp(-\alpha)}{1 - \exp(-\alpha)} \]  

where for the sake of brevity it has been assumed that \( T_s = T_c \).

By taking the limit of divergency \( \alpha \to 0 \) we obtain

for \( 0 \leq \tau \leq 1 \)
\[ \varphi(\tau) = \frac{1 - \exp\left( -\psi \tau \right)}{1 - \exp\left( -\psi \right)} \]  
for \( 1 \leq \tau \leq 2 \)
\[ \varphi(\tau) = \frac{\exp\left( -\psi(\tau - 1) \right) - \exp(-\psi)}{1 - \exp(-\psi)} \]  

while, symmetrically, for convexity \( \psi \to 0 \), we obtain

for \( 0 \leq \tau \leq 1 \)
\[ \varphi(\tau) = \frac{1 - \exp\left( -\alpha \tau \right)}{1 - \exp\left( -\alpha \right)} \]  
for \( 1 \leq \tau \leq 2 \)
\[ \varphi(\tau) = \frac{\exp\left( -\alpha(\tau - 1) \right) - \exp(-\alpha)}{1 - \exp(-\alpha)} \]  

Finally, for both \( \alpha \to 0 \) and \( \psi \to 0 \) we obtain the dimensionless outflow for a planar and rectangular hillslope, as follows:

for \( 0 \leq \tau \leq 1 \)
\[ \varphi(\tau) = \tau \]  
for \( 1 \leq \tau \leq 2 \)
\[ \varphi(\tau) = 2 - \tau \]  

Eqs. (29) and (30) are then written as follows:

for \( 0 \leq \tau \leq 1 \)
\[ a = 0: \quad \sigma(\tau) = \varphi(\tau) \left( \frac{\exp(\psi) - 1}{\exp(\psi) - 1 - \psi} \right) - \tau \left( \frac{\exp(\psi) - 1 - \psi}{\exp(\psi) - 1 - \psi} \right) \]
\[ \gamma = 0: \quad \sigma(\tau) = \varphi(\tau) \left( \frac{\exp(\psi) - 1}{\exp(\psi) - 1 - \psi} \right) - \frac{\alpha}{\exp(\psi) - 1 - \psi} \]

for \( 1 \leq \tau \leq 2 \)
\[ a = 0: \quad \sigma(\tau) = 1 + (\varphi(\tau) - 1) \left( \frac{\exp(\psi) - 1}{\exp(\psi) - 1 - \psi} \right) + (\tau - 1) \left( \frac{\psi}{\exp(\psi) - 1 - \psi} \right) \]
\[ \gamma = 0: \quad \sigma(\tau) = 1 + (\varphi(\tau) - 1) \left( \frac{\exp(\psi) - 1}{\exp(\psi) - 1 - \psi} \right) + (\tau - 1) \left( \frac{\alpha}{\exp(\psi) - 1 - \psi} \right) \]

Since both the dimensionless outflow and volume may be expressed in terms of the variables \( \alpha, \psi \) and \( \tau \), these results show that the dimensionless hysteresis depends only on the variables \( \alpha \) and \( \psi \). The dimensionless geometrical parameters \( \alpha \) and \( \psi \) define therefore the hydrological similarity between hillslopes with respect to their hysteretic effect.

We used the area of the loop in the dimensionless form of the hysteresis relationship to quantify the hysteretic behaviour. Following Hayden et al. [15] and Mishra and Seth [20], the area is expressed mathematically as

\[ \eta = \frac{1}{2} \int_0^2 \left( \phi \frac{d\sigma}{d\tau} - \sigma \frac{d\phi}{d\tau} \right) d\tau \]  

The hysteresis \( \eta \) has been therefore computed for a range of values of \( \alpha, \psi \). This relationship is plotted in terms of \( \alpha, \psi \) (Fig. 6). Fig. 6 reports also the hysteresis for the hillslopes 1–9 discussed above.

The plot generalises the findings of the previous section for the nine hillslope geometries. Slopes exhibit generally clockwise hysteretic loop in the \( V-Q \) plot. Hysteresis increases with decreasing \( \alpha \) and \( \psi \), i.e. with increasing convergence (for the shape) and concavity (for the profile). The rate of change of hysteresis is relatively large at small values of \( \alpha \) and \( \psi \), then decreases with decreasing the values of the two dimensionless variable. Hysteresis approaches 1 as either \( \alpha \) or \( \psi \) approaches negative infinity. This shows that by increasing convergence (for the shape) and concavity (for the profile) the hillslope response increases the exponential growth at later stages of the recharge, while saturated moisture content tends to increase linearly with time.

For relatively large values of \( \alpha \) and \( \psi \) the hysteresis may take negative values, i.e. the loop cycle may reverse his sign from clockwise to anticlockwise.

Note that the hysteresis in the \( \alpha - \psi \) plan defines a symmetrical pattern relative to 45° axis. A characteristic point in the plot is that resulting for the planar and rectangular hillslope, i.e. for \( \alpha = \psi = 0 \). For this case, the hysteresis \( \eta \) amounts to 0.3.
4. Determination of the hillslope similarity parameters for three natural hillslopes

The hillslope similarity parameters were determined for three hillslopes located in the experimental mountainous Vauz basin in the Eastern Italian Alps (Figs. 7 and 8). This is a very steep environment with altitude ranging between 1800 and 3100 m a.s.l. [5,23]. Application of the theory to these hillslopes provides an opportunity to examine how natural topographies are represented by the two hillslope hydrological similarity parameters.

The hillslopes have been named “Piramide”, “Emme” and “Vallecola”, after their main morphological feature, with area of 0.15, 0.37 and 0.36 ha, respectively. Topography is mainly convex (Piramide), planar (Emme) and concave (Vallecola) (Fig. 8). Average gradients range around 1:1. A detailed digital elevation model (DEM) with a 1 m resolution is available for these hillslopes. We check the plausibility of using a second order polynomial function to describe the bedrock slope and an exponential function to describe the variation of the width of the hillslope as function of distance downslope. The hillslope convergence rate $a$ was determined from the area of each hillslope $A$ and from the hillslope width at the divide $C$ based on the relationship

$$A = \frac{C}{a} (\exp aL - 1)$$  \hspace{1cm} (38)

This approach is preferable with respect to regression of hillslope width against downslope distance, since it ensures mass conservation. While regression of width against downslope distance minimizes the squared differences between observed and modelled widths, use of Eq. (38) identifies the function that, starting by the observed hillslope width at the divide (or at the base of the hillslope), has the area under the function equal to the observed one. The fit of these equations to measures of both hillslope elevation and hillslope width for the three hillslopes is described in Fig. 9, with correlation always exceeding 0.99 for the second order polynomial and ranging from 0.75 to 0.95 for the exponential width function. This shows that the polynomial function describes adequately the hillslope elevations. Hillslope width is adequately described by the exponential function for Emme and Vallecola, while a lower accuracy is demonstrated for Piramide. In the case of Piramide, width measures close to the hillslope peak are rather small compared to DEM resolution. It is likely that errors in these measures contribute to the lower accuracy demonstrated in this case. The geometric parameters of the polynomial function and of the exponential function are reported in Table 2, where the parameter $C_1$ is the measured width at the hillslope base. The values of parameter $\beta$, all around 0.9, show that the average gradient is high, close to 1:1. The range of parameter $\gamma$ is similar to that explored for the basic hillslopes analysed in Section 3, as well as the range of convergence ratios $a$. Based on these geometric parameters, the hillslope similarity parameters were computed for the three hillslope (Table 3) and plotted in Fig. 10. This figure shows how the two hillslope similarity parameters capture the essential geometric features of each hillslope. First of all, it can be seen that the three hillslopes fall within or close to the geometric space identified by the
nine basic relief forms analysed in Section 3. Both similarity parameters are positive for Piramide, with the $\alpha$ divergency parameter larger than the $\psi$ convexity parameter. This indicates that for this relief form divergency dominates over convexity. Emme is shown to be a relatively planar hillslope, with slightly negative values for both

Fig. 7. Rio Vauz catchment and localization of the three natural hillslopes.

Fig. 8. Topography of the three natural hillslopes.
similarity parameters. Negative and relatively similar parameter values are reported for Vallecola, which is therefore identified as a convergent and concave hillslope, as expected. These findings show that the two hillslope similarity parameters can be effectively used to partition the landscape into converging and diverging hillsides, defining the hydrological similarity between hillslopes with respect to their characteristic response and hysteresis.

5. Concluding remarks

We have presented general analytical solutions to the hillslope-storage kinematic wave equation for subsurface flow that allow the computation of characteristic response functions for every type of hillslopes which can be described by using a second order polynomial function for the profile and an exponential equation for the width function. The hillslope-storage subsurface flow equation takes into account the topographic controls exerted on the flow processes by the plan shape and profile curvature. Our analytical solution provides a physical basis for deriving two geometric parameters \( a \) and \( w \) which define the hydrological similarity between hillslopes with respect to their characteristic response and hysteresis. Therefore, each hillslope may be represented in a \( a - w \) plot which allows to derive essential information on its response.

The hysteresis \( \eta \), quantified by the area of the hysteretic dimensionless loop, has been therefore computed for a range of values of parameters \( a \) and \( w \). Slopes exhibit generally clockwise hysteretic loop in the flux-storage plot. It has been found that hysteresis increases with decreasing \( a \) and \( w \), i.e. with increasing convergence (for the shape) and concavity (for the profile), and vice versa. For rela-
tively large values of $a$ and $w$ the hysteresis may take a complex pattern, with combination of clockwise to anticlockwise loop cycles.

We propose that analytical solutions like those provided here offer essential insights in the functioning of hillslopes and may form the basis of hillslope similarity analysis.

The validity of our results is restricted by (1) the kinematic wave assumption of subsurface flow, (2) the assumption of specific functions for hillslope profile and width, (3) the assumption of spatially homogeneous hydraulic characteristics of the hillslopes, (4) the assumption that capillarity effects in the unsaturated zone above the phreatic layer can be neglected, (5) recharge is spatially uniform, and (6) the soil moisture capacity always exceeds the computed soil moisture. Because we assume that the kinematic wave approximation holds, the analysis presented here is limited to moderate to steep slopes. The lower limit of relevant slopes is defined by the ratio of gravity drainage versus diffusion drainage, as well as by water table gradients. Since we have formulated the kinematic form of Darcy’s law under the assumption of horizontal flow lines, there exists also an upper limit to the relevant bedrock slopes [7].

Example applications have been described for three hillslopes in the Eastern Italian Alps. These applications tested the suitability of the analytical functions to describe the plan and curvature shapes of natural hillslope and provided an opportunity to examine how natural topographies are represented by the two hillslope hydrological similarity parameters.

We emphasise that the type of hysteretic behaviour investigated here is limited to the saturated component of the hillslope response and does not include the unsaturated component. The contribution of unsaturated component will delay significantly the response [3] and as such it may modify significantly the hysteretic behaviour considered here.

It is likely that the strong hysteretic effect emerging for convergent-concave hillslopes will be limited by saturation effects when the soil moisture capacity is less than the computed soil moisture. As the water table swells, this increases the groundwater-surface water contact areas and results in enhanced outflow. This enhanced flow will reduce the hillslope storage, completing the negative feedback and bringing itself to the original state faster than otherwise [10].

Further research is being carried out to extend the analytical framework to consider the unsaturated component of the hillslope response and to validate the hillslope geometrical similarity numbers defined here as hillslope subsurface flow similarity parameter by confrontation with experimental data. Further work on hysteresis is being conducted on deriving the relationship between saturated area and subsurface outflow. Finally, our analysis has been conducted at the hillslope scale and the scaling from hillslopes to catchments deserves further investigation. This may take

Fig. 10. The hillslope similarity parameters $\alpha$ and $\psi$ for the three natural hillslopes.
the form of analysis of hysteresis of a population of individual hillslopes, each defined with its geometrical parameters.

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